BRAUER-TYPE RECIPROCITY FOR A CLASS OF GRADED ASSOCIATIVE ALGEBRAS

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Classical Brauer reciprocity can be stated roughly as follows: Let G be a finite group and let k be an algebraically closed field of characteristic $p > 0$. If S is a simple kG-module and $P(S)$ is its projective cover, then the multiplicity of a simple module L as a composition factor of a characteristic zero lift of $P(S)$ is the same as the multiplicity of S as a composition factor of a modulo p reduction of L. (All modules in this paper are assumed to be finite dimensional.) One thinks of the module L as playing an intermediate role between $P(S)$ and S.

Reciprocities similar to Brauer reciprocity (here called "Brauer-type reciprocities") have subsequently been found to occur in many other settings. For instance, if $\mathfrak{g}$ is a classical (modular) Lie algebra (definition given in §2), then there exists a set $\mathcal{Z}$ of $\mathfrak{g}$-modules with the following property: Given a simple $\mathfrak{g}$-module S, the projective cover $P(S)$ of S has a filtration with each successive quotient (isomorphic to a module) in $\mathcal{Z}$ and for each such filtration, the number of times $Z \in \mathcal{Z}$ occurs is the same as the multiplicity of S as a composition factor of Z. This was proved for $\mathfrak{g}$ of type A_1 by Pollack ([10]) and for arbitrary $\mathfrak{g}$ by Humphreys ([3]). (The reciprocity in this setting is often called "Humphreys reciprocity.")

Inspired by Humphreys' result, Bernstein, Gelfand and Gelfand sought and found a Brauer-type reciprocity ("BGG reciprocity") in a certain "truncated" category (their category "$\mathcal{O}$") of modules for a complex semisimple Lie algebra (see [1]). Later, this was generalized by Mirollo and Vilonen ([8]) to the category of perverse sheaves on a complex analytic space.

In [7] Jantzen used techniques of Bernstein, Gelfand and Gelfand to prove a Brauer-type reciprocity in the category of modules for the hyperalgebra of the nth Frobenius kernel of a semisimple algebraic group scheme.

Other settings for Brauer-type reciprocities as well as axiomatic approaches can be found in [2], [4] and [11].

In this paper we list some assumptions on a finite dimensional graded associative algebra and prove a Brauer-type reciprocity in the category of its modules as well as in the category of its graded modules. To prove that each projective module has a filtration as above, it is first shown that a projective module has the structure of a graded module. The desired filtration is then constructed in the graded category (the grading being crucial for the method used). As special cases, we recover the reciprocities of Humphreys and Jantzen and we obtain new results for finite dimensional graded restricted Lie algebras (the study of which prompted the investigations leading to this paper). Techniques from [4], [6] and [7] have been used in some of the proofs below.

1. Graded algebras and graded modules

Let $A = \sum_{i\in\mathbb{Z}} A_i$ be a finite dimensional graded algebra over a field k and let $\mathbf{G}A$ (resp. $\mathbf{G}'A$) denote the category of finite dimensional graded left (resp. right) A-modules. (For definitions and basic theory of graded rings, see [9].)

If B is a graded subalgebra of A and $N \in \operatorname{ob}\mathbf{G}B$, then $A \otimes_B N \in \operatorname{ob}\mathbf{G}A$ with the ith homogeneous component $(A \otimes_B N)_i$ defined as the k-span of all $a \otimes n$ with $a \in A_j$ and $n \in N_{i-j}$. If $M \in \operatorname{ob}\mathbf{G}A$ and $f \in \operatorname{Hom}_{\mathbf{G}B}(N, M)$, then the induced homomorphism $\bar{f} : A\otimes_B N \to M$ given by $\bar{f}(a\otimes n) = af(n)$ is graded.

Let $M \in \operatorname{ob}\mathbf{G}A$. Then $M^* := \operatorname{Hom}_k(M, k)$ is an object of $\mathbf{G}'A$ with the definitions $(fa)(m) = f(am)$ ($a \in A$, $f \in M^*$, $m \in M$) and $(M^*)_i = \{\, f \in M^* \mid f(M_j) = 0 \text{ for all } j \neq -i \,\}$.

If A has an antiautomorphism t which is antigraded (meaning $t(A_i) \subseteq A_{-i}$), then the vector space M^* becomes a graded *left* A-module, denoted M^t, with the definitions $(af)(m) = f(t(a)m)$ $(a \in A,\ f \in M^t,\ m \in M)$ and $(M^t)_i = \{\, f \in M^t \mid f(M_j) = 0 \text{ for all } j \neq i \,\}$.

The *ith suspension* $M(i)$ of M is by definition the A-module M with new grading $(M(i))_j = M_{i+j}$.

Let $\mathbf{M}A$ (resp. $\mathbf{M}'A$) denote the category of left (resp. right) A-modules. The forgetful functor from $\mathbf{G}A$ to $\mathbf{M}A$ (resp. $\mathbf{G}'A$ to $\mathbf{M}'A$) will be denoted F. Since the projective objects of $\mathbf{G}A$ are precisely the summands of free objects (= direct sums of various suspensions of A), it follows that FP is projective whenever P is.

Since several of the results (and their proofs) below are valid for both graded and non-graded modules, it will be convenient to refer to either by using the notation $\mathbf{C}A$ (resp. $\mathbf{C}'A$) with $\mathbf{C}$ a fixed, but arbitrary, element of $\{\mathbf{G}, \mathbf{M}\}$. For $M, S \in \operatorname{ob} \mathbf{C}A$ with S simple, $(M : S)$ will denote the multiplicity of S as a composition factor of M.

The following slight variations on standard results will be needed.

1.1 Theorem. *Let B be a graded subalgebra of A and assume that A is flat as an object of both $\mathbf{C}B$ and $\mathbf{C}'B$. Let $M, M' \in \operatorname{ob} \mathbf{C}A$ and let $N \in \operatorname{ob} \mathbf{C}B$. For each $n \in \mathbb{Z}^+$ we have*

(1) $\operatorname{Ext}^n_{\mathbf{C}A}(M, M') \cong \operatorname{Ext}^n_{\mathbf{C}'A}(M'^*, M^*)$,
(2) $\operatorname{Ext}^n_{\mathbf{C}A}(A \otimes_B N, M) \cong \operatorname{Ext}^n_{\mathbf{C}B}(N, M)$ *and*
(3) $\operatorname{Ext}^n_{\mathbf{C}A}(M, (N^* \otimes_B A)^*) \cong \operatorname{Ext}^n_{\mathbf{C}B}(M, N)$

and the isomorphisms are natural in the variables M, M' and N.

Proof. The standard proofs of (1) and (2) carry over to the graded situation. For (3), use (1) and (2) to get

$$\begin{aligned}\operatorname{Ext}^n_{\mathbf{C}A}(M, (N^* \otimes_B A)^*) &\cong \operatorname{Ext}^n_{\mathbf{C}'A}(N^* \otimes_B A, M^*)\\ &\cong \operatorname{Ext}^n_{\mathbf{C}'B}(N^*, M^*)\\ &\cong \operatorname{Ext}^n_{\mathbf{C}B}(M, N),\end{aligned}$$

as desired. □

2. Assumptions

The following will be assumed for the rest of the paper.

2.1. $A = \dot{\sum}_{i \in \mathbb{Z}} A_i$ *is a finite dimensional graded (associative) algebra (with identity) over an algebraically closed field k with graded subalgebras (containing 1_A) $A^- \subseteq k \cdot 1_A + \sum_{i<0} A_i$, $A^0 \subseteq A_0$ and $A^+ \subseteq k \cdot 1_A + \sum_{i>0} A_i$ such that*

(i) $A = A^- A^0 A^+$,
(ii) $\dim_k A = \dim_k A^- \dim_k A^0 \dim_k A^+$ *and*
(iii) $A^- A^0 = A^0 A^-$ *and* $A^0 A^+ = A^+ A^0$.

Examples. 1. An arbitrary finite dimensional algebra A over k possesses a trivial grading $A_0 = A$ and, with the assignments $A^0 = A_0$ and $A^- = k \cdot 1_A = A^+$, satisfies (2.1). This is a somewhat trivial example and the findings of this paper give no information about A in this case. However, it is still a good example to keep in mind when trying to formulate properties that hold for arbitrary A satisfying (2.1).

2. Let A be as in (2.1). Assume A has a graded subalgebra B and set $B^- = A^- \cap B$, $B^0 = A^0 \cap B$ and $B^+ = A^+ \cap B$. If either $B = B^- B^0 B^+$ or $\dim_k B = \dim_k B^- \dim_k B^0 \dim_k B^+$, then B satisfies (2.1) (with A replaced by B).

For the remaining examples, k has characteristic $p > 0$.

3. Let $\mathfrak{g} = \sum_{i \in \mathbb{Z}} \mathfrak{g}_i$ be a finite dimensional graded restricted Lie algebra over k. The restricted enveloping algebra A of $\mathfrak{g}$ is, by definition, the quotient of the universal enveloping algebra of $\mathfrak{g}$ by the ideal generated by all $X^p - X^{[p]}$ with $X \in \mathfrak{g}$. If $\mathcal{X}_i$ is a basis for $\mathfrak{g}_i$, then A has a basis consisting of the cosets of products of the form $\prod_i \prod_{X \in \mathcal{X}_i} X^{n(i,X)}$, with $0 \leq n(i, X) < p$, where the second product is with respect to a fixed ordering on $\mathcal{X}_i$ (see [5]). Let A_i be the k-span of all those basis elements with $\sum_{j,X} n(j, X)j = i$. Then the A_i are homogeneous components for a grading on A. Furthermore, if A^-, A^0 and A^+ are the k-spans of

those basis elements with $n(i, X) = 0$ for all $i \geq 0$, $i \neq 0$ and $i \leq 0$, respectively, then with these definitions, A satisfies (2.1).

4. Let $\mathfrak{g}_{\mathbb{C}}$ be a simple finite dimensional complex Lie algebra. If Φ is the set of roots of $\mathfrak{g}_{\mathbb{C}}$ relative to a fixed Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$, then $\mathfrak{g}_{\mathbb{C}}$ has a basis $\{ X_\alpha, H_i \mid \alpha \in \Phi,\ 1 \leq i \leq \dim_{\mathbb{C}} \mathfrak{h}_{\mathbb{C}} \}$ (a "Chevalley basis") the $\mathbb{Z}$-span $\mathfrak{g}_{\mathbb{Z}}$ of which is closed under the bracket product. The vector space $\mathfrak{g}_{\mathbb{Z}} \otimes k$ with the induced bracket product is a (restricted) Lie algebra called a "classical Lie algebra." Set $\mathfrak{g}_0 = \mathfrak{h}_{\mathbb{Z}} \otimes k$, where $\mathfrak{h}_{\mathbb{Z}}$ is the $\mathbb{Z}$-span of the H_i, and let $\mathfrak{g}_i$ ($i \neq 0$) be the k-span of all $X_\alpha \otimes 1$ with $ht(\alpha) = i$, where $ht(\alpha)$ is the height of α relative to a fixed choice of simple roots. Then $\mathfrak{g}$ is a graded Lie algebra with ith homogeneous component $\mathfrak{g}_i$. The preceding example shows how to define a structure on the restricted enveloping algebra of $\mathfrak{g}$ in such a way that (2.1) is satisfied.

5. A finite dimensional restricted Lie algebra of Cartan type (or more generally, the p-hull of a not necessarily restricted such algebra) has a natural grading which (by example (3) again) yields a structure on the restricted enveloping algebra satisfying (2.1). (For the definition and properties of Lie algebras of Cartan type, see [12] or [13].)

6. Let A be the hyperalgebra of the nth Frobenius kernel of a semisimple algebraic group scheme over k. This algebra has a basis consisting of monomials of the form

$$\prod_{\alpha \in \Phi^-} X_{\alpha, r_\alpha} \prod_{j=1}^{\ell} H_{j, t_j} \prod_{\beta \in \Phi^+} X_{\beta, r_\beta}$$

with $0 \leq r_\gamma, t_j < p^n$. Here Φ^+ (resp. Φ^-) is the set of positive (resp. negative) roots with respect to a fixed set of simple roots Δ and the products are with respect to a fixed ordering. Set $\Phi = \Phi^- \cup \Phi^+$ and let A_i be the k-span of all monomials with $\sum_{\gamma \in \Phi} r_\gamma ht(\gamma) = i$ where ht is the height function on Φ relative to Δ. Then A is a graded algebra with ith homogeneous component A_i. Moreover, if A^0 is the k-span of those monomials with $r_\gamma = 0$ for all $\gamma \in \Phi$ and A^+ (resp. A^-) is the k-span of those monomials with $t_j = 0$ for all j and $r_\gamma = 0$ for all $\gamma \in \Phi^-$ (resp. $\gamma \in \Phi^+$), then A, with these definitions, satisfies (2.1).

3. Simple objects of $\mathbf{C}A$ and their projective covers

Set $B^+ = A^0 A^+$ and $B^- = A^- A^0$. By assumption (2.1(iii)) these are (graded) subalgebras of A.

3.1 Lemma. *If $M \in \operatorname{ob} \mathbf{C}B^+$, then $A \otimes_{B^+} M \cong B^- \otimes_{A^0} M$ in $\mathbf{C}B^-$.*

Proof. The $\mathbf{C}A^0$-isomorphism $M \to 1 \otimes M \subseteq A \otimes_{B^+} M$ induces a $\mathbf{C}B^-$-morphism $B^- \otimes_{A^0} M \to A \otimes_{B^+} M$ which is clearly surjective. By assumptions (2.1(i) and (ii)) A is a free right B^+-module and B^- is a free right A^0-module, the rank of each being $\dim_k A^-$. Therefore, $\dim_k A \otimes_{B^+} M = \dim_k A^- \dim_k M = \dim_k B^- \otimes_{A^0} M$ and the lemma follows. □

The sets $N^+ = \sum_{i>0} (B^+)_i$ and $N^- = \sum_{i<0} (B^-)_i$ are graded ideals of B^+ and B^-, respectively. Since $B^+/N^+ \cong A^0 \cong B^-/N^-$, the category $\mathbf{C}A^0$ embeds in $\mathbf{C}B^+$, as well as in $\mathbf{C}B^-$, as a full subcategory. Moreover, since N^+ and N^- are nilpotent, the simple objects of these three categories coincide. Let $\Lambda_{\mathbf{C}}$ be a fixed set of isomorphism class representatives of these simple objects.

For each $\lambda \in \Lambda_{\mathbf{C}}$, define $M(\lambda) = A \otimes_{B^+} \lambda$. (3.1) and (1.1) imply that, for each $\mu \in \Lambda_{\mathbf{C}}$,

$$\operatorname{Hom}_{\mathbf{C}B^-}(M(\lambda), \mu) \cong \operatorname{Hom}_{\mathbf{C}B^-}(B^- \otimes_{A^0} \lambda, \mu) \cong \operatorname{Hom}_{\mathbf{C}A^0}(\lambda, \mu)$$

which is isomorphic to k if $\lambda \cong \mu$ and zero otherwise. It follows that $M(\lambda)$ has a unique simple quotient $L(\lambda)$ in $\mathbf{C}A$.

3.2 Lemma. *$\{L(\lambda) \mid \lambda \in \Lambda_{\mathbf{C}}\}$ is a complete set of pairwise nonisomorphic simple objects of $\mathbf{C}A$.*

Proof. Let S be a simple object of $\mathbf{C}A$. For some $\lambda \in \Lambda_{\mathbf{C}}$ there exists a $\mathbf{C}B^+$-monomorphism $\lambda \to S$. The image of the induced $\mathbf{C}A$-morphism $M(\lambda) = A \otimes_{B^+} \lambda \to S$ is a nonzero subobject of S and hence equal to S; that is, $S \cong L(\lambda)$.

Since $L(\lambda)$ has unique simple quotient λ in $\mathbf{C}B^-$ (as follows from the paragraph preceding the lemma), the objects are pairwise nonisomorphic. □

For $\nu \in \Lambda_{\mathbf{M}}$ and $i \in \mathbb{Z}$, let $\nu(i)$ denote the graded A^0-module with underlying A^0-module ν and grading $\nu(i)_j = \delta_{ij}\nu$. Evidently, $\{\nu(i) \mid \nu \in \Lambda_{\mathbf{M}}, i \in \mathbb{Z}\}$ is a complete set of pairwise nonisomorphic simple graded A^0-modules. Therefore, by adjusting the choice of isomorphism class representatives if necessary, it can be assumed that $\Lambda_{\mathbf{G}} = \{\nu(i) \mid \nu \in \Lambda_{\mathbf{M}}, i \in \mathbb{Z}\}$. In particular, the forgetful functor maps $\Lambda_{\mathbf{G}}$ onto $\Lambda_{\mathbf{M}}$.

Denote the projective cover (in $\mathbf{C}A^0$) of $\lambda \in \Lambda_{\mathbf{C}}$ by $P(\lambda)$. For each $\lambda \in \Lambda_{\mathbf{G}}$, $P(\lambda)$ is indecomposable and hence it has a unique nonzero homogeneous component as each homogeneous component is an A^0-submodule. Therefore, $FP(\lambda)$ is also indecomposable and projective (see §1), whence $FP(\lambda) \cong P(F\lambda)$.

Denote the projective cover of $M \in \operatorname{ob}\mathbf{C}A$ by $P(M)$.

3.3 Theorem. *For each $\lambda \in \Lambda_{\mathbf{G}}$, $FL(\lambda) \cong L(F\lambda)$ and $FP(L(\lambda)) \cong P(L(F\lambda))$.*

Proof. First note that $M(\mu(i)) \cong M(\mu)(i)$ and $FM(\mu) \cong M(F\mu)$ for each $\mu \in \Lambda_{\mathbf{G}}$ and $i \in \mathbb{Z}$. Therefore, $L(\mu(i)) \cong L(\mu)(i)$ and $L(F\mu)$ is a homomorphic image of $FL(\mu)$.

Now, for each $\lambda \in \Lambda_{\mathbf{C}}$, set $I(\lambda) = A \otimes_{A^0} P(\lambda)$. By (1.1), the functor $\operatorname{Hom}_{\mathbf{C}A}(I(\lambda), \cdot)$ is naturally isomorphic to the functor $\operatorname{Hom}_{\mathbf{C}A^0}(P(\lambda), \cdot)$ which is exact, implying that $I(\lambda)$ is projective. Moreover, for each $\lambda \in \Lambda_{\mathbf{C}}$ and $L \in \operatorname{ob}\mathbf{C}A$,

$$\dim_k \operatorname{Hom}_{\mathbf{C}A}(I(\lambda), L) = \dim_k \operatorname{Hom}_{\mathbf{C}A^0}(P(\lambda), L) = (L : \lambda).$$

Therefore, for each $\lambda \in \Lambda_{\mathbf{G}}$,

$$\begin{aligned}
\sum_{\mu \in \Lambda_{\mathbf{G}}} \dim_k \operatorname{Hom}_{\mathbf{G}A}(I(\lambda), L(\mu)) &= \sum_{\mu} (L(\mu) : \lambda) \\
&= \sum_{\substack{\nu \in \Lambda_{\mathbf{M}} \\ i \in \mathbb{Z}}} (L(\nu(i)) : \lambda) \\
&= \sum_{\nu, i} (L(\nu(0)) : \lambda(-i)) \\
&= \sum_{\nu} (FL(\nu(0)) : F\lambda) \\
&\geq \sum_{\nu} (L(\nu) : F\lambda) \\
&= \sum_{\nu \in \Lambda_{\mathbf{M}}} \dim_k \operatorname{Hom}_{\mathbf{M}A}(I(F\lambda) : L(\nu)).
\end{aligned}$$

The first sum counts the number of indecomposable summands of $I(\lambda)$ while the last counts that of $I(F\lambda) \cong FI(\lambda)$. Generally, the former is less than or equal to the latter, so the computation shows that these numbers are in fact equal. In particular, $(FL(\nu(0)) : F\lambda) = (L(\nu) : F\lambda)$ for each $\nu \in \Lambda_{\mathbf{M}}$, $\lambda \in \Lambda_{\mathbf{G}}$. Thus, if $\mu \in \Lambda_{\mathbf{G}}$, then $\mu = \nu(i)$ for some $\nu \in \Lambda_{\mathbf{M}}$, $i \in \mathbb{Z}$, and $FL(\mu) = FL(\nu(i)) \cong FL(\nu(0))(i) \cong L(\nu) = L(F\mu)$. This proves the first statement.

For the second statement observe that

$$\operatorname{Hom}_{\mathbf{G}A}(I(\lambda), L(\lambda)) \cong \operatorname{Hom}_{\mathbf{G}A^0}(P(\lambda), L(\lambda)) \neq 0$$

which implies that $P(L(\lambda))$ is a summand of $I(\lambda)$. The computation above shows that $FP(L(\lambda))$ is indecomposable, and since this module has $FL(\lambda)$ as a homomorphic image the theorem follows. □

4. Z-filtrations

For $\lambda \in \Lambda_{\mathbf{C}}$ set $Z(\lambda) = A \otimes_{B^+} P(\lambda)$ and $Z^-(\lambda) = (\lambda^* \otimes_{B^-} A)^*$.

4.1 Lemma. *$Z(\lambda)$ is the projective cover in $\mathbf{C}B^-$ of $\lambda \in \Lambda_{\mathbf{C}}$.*

Proof. By (3.1), $Z(\lambda)$ is projective when viewed as an object of $\mathbf{C}B^-$. Moreover, for each $\mu \in \Lambda_{\mathbf{C}}$, (1.1) gives

$$\operatorname{Hom}_{\mathbf{C}B^-}(Z(\lambda), \mu) \cong \operatorname{Hom}_{\mathbf{C}A^0}(P(\lambda), \mu)$$

which is isomorphic to k if $\lambda \cong \mu$ and zero otherwise. Therefore, λ is the unique simple quotient of $Z(\lambda)$ as required. □

4.2 Lemma. $\mathrm{Ext}^n_{\mathbf{C}A}(Z(\lambda), Z^-(\mu))$ *is isomorphic to* k *if* $n = 0$ *and* $\lambda \cong \mu$*, and it is zero otherwise.*

Proof. By (1.1) $\mathrm{Ext}^n_{\mathbf{C}A}(Z(\lambda), Z^-(\mu)) \cong \mathrm{Ext}^n_{\mathbf{C}B^-}(Z(\lambda), \mu)$. If $n > 0$ this space is zero as $Z(\lambda)$ is projective in $\mathbf{C}B^-$ by (4.1). If $n = 0$ this space is isomorphic to $\mathrm{Hom}_{\mathbf{C}B^-}(B^- \otimes_{A^0} P(\lambda), \mu) \cong \mathrm{Hom}_{\mathbf{C}A^0}(P(\lambda), \mu)$ (see (3.1)) and the lemma follows. □

An object M of $\mathbf{C}A$ is said to have a *Z-filtration* if it has a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_s = M$ by subobjects M_j such that for each $j > 0$, $M_j/M_{j-1} \cong Z(\lambda)$ for some $\lambda \in \Lambda_{\mathbf{C}}$. The following corollary shows that the number $[M : Z(\lambda)] := |\{j \mid M_j/M_{j-1} \cong Z(\lambda)\}|$ is independent of the choice of Z-filtration.

4.3 Corollary. *If* $M \in ob\,\mathbf{C}A$ *has a Z-filtration, then (with the notation as above)*

$$[M : Z(\lambda)] = \dim_k \mathrm{Hom}_{\mathbf{C}A}(M, Z^-(\lambda))$$

($\lambda \in \Lambda_{\mathbf{C}}$*).*

Proof. This follows easily from (4.2) by using induction on s. □

As a consequence of (4.1), any object of $\mathbf{C}A$ having a Z-filtration is necessarily projective when viewed as an object of $\mathbf{C}B^-$. The next theorem shows that the converse holds if $\mathbf{C} = \mathbf{G}$.

4.4 Theorem. *If an object of* $\mathbf{G}A$ *is projective when viewed as an object of* $\mathbf{G}B^-$*, then it has a Z-filtration.*

Proof. Assume that $M \in \mathrm{ob}\,\mathbf{G}A$ is projective in $\mathbf{G}B^-$. It follows from the fact that B^- is a free object of $\mathbf{G}A^0$ (see (2.1)) that M, and hence each M_i, is projective in $\mathbf{G}A^0$. Choose i maximal with $M_i \neq 0$ and write $M_i = P \dot{+} Q$ where $P \cong P(\lambda)$ for some $\lambda \in \Lambda_{\mathbf{G}}$. Let $\varphi : Z(\lambda) \to M$ denote the graded A-homomorphism induced by the graded B^+-isomorphism $P(\lambda) \xrightarrow{\sim} P \subseteq M$ (by maximality of i, $N^+P = \{0\}$ so this is indeed a B^+-homomorphism). Now $M' := Q + \sum_{j<i} M_j$ is a graded B^--submodule of M so the canonical map $\pi : M \to M/M' \cong P(\lambda)$ is a graded B^--homomorphism. Since M is projective in $\mathbf{G}B^-$, π factors through the canonical epimorphism $Z(\lambda) \to Z(\lambda)/N^-Z(\lambda) \cong P(\lambda)$ to give a graded B^--homomorphism $\psi : M \to Z(\lambda)$.

It is easy to see that $\psi\varphi \in \mathrm{End}_{\mathbf{G}B^-}(Z(\lambda))$ takes the ith homogeneous component $1 \otimes P(\lambda)$ of $Z(\lambda)$ onto itself and since this component generates $Z(\lambda)$ as a B^--module it follows that $\psi\varphi$ is surjective and hence bijective. In particular, φ is injective so that $\mathrm{im}\,\varphi \cong Z(\lambda)$ in $\mathbf{G}A$. Moreover, $M/\mathrm{im}\,\varphi$ is isomorphic in $\mathbf{G}B^-$ to a direct (graded) summand of M and is therefore projective in $\mathbf{G}B^-$. The theorem now follows by induction on $\dim_k M$. □

The desired Brauer-type reciprocity for the category $\mathbf{C}A$ is now obtained by assembling results.

4.5 Theorem. *Any projective object of* $\mathbf{C}A$ *has a Z-filtration. In particular, for each simple object S of* $\mathbf{C}A$*, the projective cover $P(S)$ of S has a Z-filtration and* $[P(S) : Z(\lambda)] = (Z^-(\lambda) : S)$ *for each* $\lambda \in \Lambda_{\mathbf{C}}$*.*

Proof. By (3.3), any projective object of $\mathbf{M}A$ is of the form FP where P is a projective object of $\mathbf{G}A$. Moreover, for each $\lambda \in \Lambda_{\mathbf{G}}$, $FZ(\lambda) \cong Z(F\lambda)$, so that FP has a Z-filtration if P does. Hence, for the first statement it may be assumed that $\mathbf{C} = \mathbf{G}$. Now, any projective object of $\mathbf{G}A$ is projective in $\mathbf{G}B^-$ as A is a free object of $\mathbf{G}B^-$ (a consequence of (2.1)), so the first statement follows from (4.4).

Finally, (4.3) applies and the proof is complete. □

5. A Special Case

The presence of the two *different* intermediate modules $Z(\lambda)$ and $Z^-(\lambda)$ in the reciprocity formula of (4.5) produces an asymmetry which is not found in many Brauer-type reciprocity formulas. In the following proposition a reciprocity involving a single intermediate module is obtained under the assumption of additional constraints on A.

5.1 Theorem. *Assume that, in addition to (2.1), A satisfies the following:*

1. A^0 *is semisimple and*
2. *A has an antigraded antiautomorphism t of order two such that* $t(B^+) = B^-$ *and* $\lambda^t \cong \lambda$ *for each* $\lambda \in \Lambda_{\mathbf{M}}$.

Then $[P(S) : Z(\lambda)] = (Z(\lambda) : S)$ *for each* $\lambda \in \Lambda_{\mathbf{C}}$ *and each simple* $S \in \operatorname{ob} \mathbf{C}A$.

Remark. Since $t(A^0) = t((B^+)_0) = (B^-)_0 = A^0$, t restricts to an antiautomorphism of A^0 so that the module λ^t is defined.

Proof. The first step is to show that $S^t \cong S$ for each simple object S of $\mathbf{C}A$, and for this it may be assumed, by (3.3) and the fact that the forgetful functor F maps $\Lambda_{\mathbf{G}}$ onto $\Lambda_{\mathbf{M}}$, that $\mathbf{C} = \mathbf{G}$ and $S = L(\lambda)$ for some $\lambda \in \Lambda_{\mathbf{G}}$. Since $M \mapsto M^t$ is an exact functor, S^t is simple, so $S^t \cong L(\mu)$ for some $\mu \in \Lambda_{\mathbf{G}}$.

Now, if $\nu \in \Lambda_{\mathbf{G}}$, then ν has a single nonzero homogeneous component, say $\nu = \nu_i$, and from the construction of the simple module $L(\nu)$, it follows that $L(\nu)_j = \{0\}$ for $j > i$ and that $L(\nu)_i \cong \nu$ in $\mathbf{G}A^0$. Consequently, if $\lambda = \lambda_i$, then $\lambda \cong \lambda^t \cong (S^t)_i \cong L(\mu)_i \cong \mu$ in $\mathbf{G}A^0$. Hence $\mu = \lambda$ and $S^t \cong S$.

Next, it is easy to check that the map $\varphi : (\lambda^* \otimes_{B^-} A)^* \to (A \otimes_{B^+} \lambda^t)^t$ given by $\varphi(f)(a \otimes \alpha) = f(\alpha \otimes t(a))$ $(f \in (\lambda^* \otimes_{B^-} A)^*,\ a \in A,\ \alpha \in \lambda^t)$ is a well-defined $\mathbf{C}A$-monomorphism. Furthermore, the dimension of the first module is $\dim_k \lambda \dim_k A^+$ while that of the second is $\dim_k \lambda \dim_k A^-$. Since these dimensions are the same, φ is an isomorphism. Finally, $P(\lambda) \cong \lambda$ by semisimplicity of A^0, so that $Z^-(\lambda) = (\lambda^* \otimes_{B^-} A)^* \cong (A \otimes_{B^+} \lambda^t)^t \cong (A \otimes_{B^+} P(\lambda))^t \cong (Z(\lambda))^t$. The first step together with (4.5) now finishes the proof of the proposition. □

(5.1) can be applied to the situation in example (4) of §2 to recover results of Humphreys in [3]. Here, A^0 is the restricted enveloping algebra of the Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$ (so A^0 is semisimple), t is the antiautomorphism of A induced by $H_i \otimes 1 \mapsto H_i \otimes 1$ and $X_\alpha \otimes 1 \mapsto X_{-\alpha} \otimes 1$, $\Lambda_{\mathbf{M}} = \mathfrak{h}^*$ (where $\lambda \in \mathfrak{h}^*$ is identified with the one-dimensional A^0-module induced by λ), and $Z(\lambda)$ is Humphreys' "standard cyclic module Z_λ."

Similarly, specializing to example (6) of §2 (which is analogous to example (4)) one can recover results of Jantzen in [7].

It should be pointed out that not all the algebras in the examples of §2 satisfy the additional conditions of the theorem. For instance, in example (5) if A is the restricted enveloping algebra of the Witt algebra $W(1,1)$ (endowed with the usual grading) then $\dim_k B^+ = p^{p-1}$ while $\dim_k B^- = p^2$.

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